Math 241

Problem Set 10 solution manual

Exercise. A10.1

i $G = S_4$, then $|G| = 4! = 24 = 2^3.3$.
A sylow 2-subgroup of S_4 is a subgroup of order 8, so we can consider D_4.
A sylow 3-subgroup of S_4 is a subgroup of order 3, so we can consider the subgroup generated by any 3-cycle. eg: $<(123)>$.

ii $G = S_5$, then $|G| = 5! = 120 = 2^3.3.5$.
A sylow 2-subgroup of S_5 is a subgroup of order 8. Notice that the elements of D_4 when considered as pemutations in S_5 will still form a subgroup whose order would be 8.
A sylow 3-subgroup of S_5 is a subgroup of order 3, so we can consider the subgroup generated by any 3-cycle.
A sylow 5-subgroup of S_5 is a subgroup of order 5, so we can consider the subgroup generated by any 5-cycle.

iii $G = \mathbb{Z}_{100} \times \mathbb{Z}_{30}$, then $|G| = 3000 = 2^3.3.5^3$
A sylow 3-subgroup of $\mathbb{Z}_{100} \times \mathbb{Z}_{30}$ is a subgroup of order 3, so we can consider the subgroup $<(0,10)>=\{(0,0),(0,10),(0,20)\}$.
A sylow 2-subgroup of $\mathbb{Z}_{100} \times \mathbb{Z}_{30}$ is a subgroup of order 8, so we can consider the subgroup $<(25,0),(0,15)>=\{(0,0),(25,0),(50,0),(75,0),(0,15),(25,15),(50,15),(75,15)\}$.
A sylow 5-subgroup of $\mathbb{Z}_{100} \times \mathbb{Z}_{30}$ is a subgroup of order 125, so we can consider the subgroup $<(4,0),(0,6)>=\{(a,b) \mid a \equiv 0\ mod(4), \text{ and } b \equiv 0\ mod(6)\}$.

iv $G = GL_2(\mathbb{Z}_7)$, then $|G| = (7^2-1)(7^2-7) = 2016 = 2^5.3^2.7$.
A sylow 7-subgroup of G is a subgroup of order 7, so we can consider the subgroup:
$$\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}\} =< \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} >.$$

A sylow 3-subgroup of G is a subgroup of order 9, so we can consider the subgroup:
$$\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a,b \in \{1,2,4\}\}.$$

A sylow 2-subgroup of G is a subgroup of order 2^5, so we can consider the subgroup:
So consider first the subgroup $H' =< \begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix} >$, H' is a subgroup of order 16. Then our Sylow subgroup is $H = H' \cup \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} H'$

Exercise. A10.2

a- First we find the normalizer of T. So let $g = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \in GL(2,\mathbb{R})$, for g to be in the normalizer of T then we must have $g.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.g^{-1} = \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} \implies \{a', b'\} = \{1, 2\}$, reason is because the matrix $\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}$ must have 1,2 as eigen values.

Then we have the following : either $g.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.g^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ which implies that $g.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} g \implies \begin{bmatrix} i & 2j \\ k & 2l \end{bmatrix} = \begin{bmatrix} i & j \\ 2k & 2l \end{bmatrix} \implies j = k = 0 \implies g$ must be a diagonal matrix $\implies g \in T$

or $g.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.g^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ which implies that $g.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} g \implies \begin{bmatrix} i & 2j \\ k & 2l \end{bmatrix} = \begin{bmatrix} 2i & 2j \\ k & l \end{bmatrix} \implies i = l = 0 \implies g \in \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T$

This show that $N(T) \subseteq T \cup \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T$, Now it is easy to check that the right side is also contained in the normalizer of T.
Now we can deduce that index of T in $N(T)$ is 2.

Now For the normalizer of U, we do the same, you can start by finding the elements g such that $g\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} g^{-1} \in U$, and you will get that g must be either in U or in $\begin{bmatrix} 1 & 0 \\ 0 & -l \end{bmatrix} U$.
FInally you can verify that $U \cup \begin{bmatrix} 1 & 0 \\ 0 & -l \end{bmatrix} U \subset N(U)$ and hence $N(U) = U \cup \begin{bmatrix} 1 & 0 \\ 0 & -l \end{bmatrix} U$.
Then we can deduce that index of U in $N(U)$ is 2.

b- Now to find the normalizer for T consider the matrix $A = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & n \end{bmatrix}$,

$g \in N(T) \implies gAg^{-1} = \begin{bmatrix} a_1 & 0 & \dots & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & a_n \end{bmatrix}$ but this implies that $\{a_1, a_2, ..., a_n\} = \{1, 2, ..., n\}$ in some order. So $\exists\ \pi \in S_n$ such that $a_i = \pi(i)\ \forall i \in \{1, 2, ..., n\}$.

So $gAg^{-1} = \begin{bmatrix} \pi(1) & 0 & \dots & \dots & 0 \\ 0 & \pi(2) & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & \pi(n) \end{bmatrix} = P_\pi \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & 0 & \ddots & 0 \\ 0 & \dots & \dots & 0 & n \end{bmatrix} P_\pi^{-1}$ where P_π is a permutation matrix.

so we get that $g \in P_\pi.Stab(A) = P_\pi.T$ for some $\pi \in S_n$.

Hence $N(T) = \bigcup_{\pi \in S_N} P_\pi.T$, and then $N(T)/T = \{\bar{P}_\pi \mid \pi \in S_n\} \cong S_n$.

Section. 37

Exercise. 5

Let G be a group of order 96. G must contain a sylow 2-subgroup of order 2^5. If G containes 1 sylow 2-subgroup then we know that this subgroup is normal and hence G is not simple.

So consider that G containes more than one sylow 2-subgroups H and K. $H \cap K$ is a subgroup if H and of K $(H \neq K)$, by a counting argument we can see that the order of $H \cap K$ is 16. Moreover, the normalizer of $H \cap K$ containes both subgroups H, and K, hence its order should be a divisor of 94 which is greater than 32, hence $N(H \cap K) - G$, so $H \cap K$ is normal in G.

Section. 18

Exercise. 11

It is easy to see that this set is closed under addition and multicplication, for example $(a+b\sqrt{2})(a'+b'\sqrt{2}) = aa' + 2bb' + \sqrt{2}(ab' + ba')$ with $aa' + 2bb', ab' + ba' \in \mathbb{Z}$. Moreover, 0=0+0$\sqrt{2}$ and 1=1+0$\sqrt{2}$ both belong to the set. Assotiativity and distributativity follows directly from the properties of the usual addition and multiplication in $\mathbb{R}$. So we have a ring. This ring is commutative since usual multiplication is commutative.

This ring is not a field since 2 is not a unit, for if it was then $\exists\ x \in R$ such that $2x = 1$, then $x = 1/2$ contradiction ($1/2 = a + b\sqrt{2}$ with $a, b \in \mathbb{Z}$ then $\sqrt{2} = \frac{1/2-a}{b}$ which is impossibe.

Exercise. 12

Similarly as above it is easy to see that this set is a commutative ring. Now let $x_1 = a + b\sqrt{2}$ be an non-zero element in the ring , notice that then $a^2 - 2b^2 \neq 0$ since $a, b \in \mathbb{Q}$,and then the element $x_2 = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$ is an element in the ring with the property $x_1.x_2 = 1$. Then our ring is a field.

Exercise. 15

The units in $\mathbb{Z} \times \mathbb{Z}$ are : (1,1), (-1,1), (1,-1), (-1,-1).

Exercise. 20

a- The number of elements in $M_2(\mathbb{Z}_2)$ is 16, since each entery of the matrix can be either 0, or 1.

b- The invertible elements are: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Exercise. 38

R is commutative $\Leftrightarrow ab = ba \ \forall\, a, b \in R \Leftrightarrow ab - ba = 0 \ \forall\, a, b \in R \Leftrightarrow a^2 - b^2 + (ab - ba) = a^2 - b^2$ $\Leftrightarrow (a - b)(a + b) = a^2 - b^2$.